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q-deformation of radial problems: the simple harmonic oscillator in two dimensions

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Abstract. The possibility of q-deformation of known exactly solvable quantum mechanical models is considered via the dynamical symmetry of the radial problem. The algebraic structure of the D-dimensional simple harmonic oscillator is analysed from this point of view. In the two-dimensional case, q-deformed bosons are introduced for the radial degrees of freedom, which together with the standard angular operators yield an $\mathrm{so}(2) \oplus \mathrm{so}_q(2,1)$ dynamical algebra. The corresponding Hilbert space is identified and possible q-deformed Hamiltonians described.

1. Introduction and main results

The subject of quantum groups, or quantum universal enveloping algebras has received recent impetus from its deep applications in exactly solvable models in statistical mechanics [1], conformal field theory, and two-dimensional systems with intermediate statistics [2]. Their natural emergence as generalized symmetries of physical systems has engendered studies of, for example, their possible rôle as gauge symmetries [3].

In quantum mechanical problems, there are several ways in which 'q-deformations' can be addressed. The most ambitious connects with non-commutative geometry [4] and, at the level of the Schrödinger equation, uses the apparatus of q-differential calculus [5]: within such a programme, q-deformed symmetry algebras would emerge directly from q-differential realizations via special choices of potential, magnetic field, and the like. More algebraically, one can postulate suitable operators such as q-deformed bosons or fermions [6, 7, 8], and ask what q-symmetries are realized in terms of them for suitable Hamiltonians [9], or conversely what Hamiltonians will lead to given q-symmetries [10, 11]. In this approach there appear to be some difficulties in making ordinary dynamics, as expressed by the Heisenberg equations of motion, compatible with the q-commutator structures of the postulated oscillators [9].

At the level which probes least the details of the underlying dynamics, the kinematical degeneracy algebra, or dynamical spectrum generating algebra, may simply be replaced by the appropriate q-deformed algebra. Thus, for example, for molecular rigid rotors one considers $U_q(\mathrm{su}(2))$, [12]; for the Jaynes–Cummings model in quantum optics, $U_q(\mathrm{so}(2,1))$ [13]; and for the 'quantum hydrogen atom', forms of $U_q(\mathrm{so}(4))$ [14]. In these examples, typically the Hamiltonian, in terms of generators

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or invariants of the symmetry algebra, is transcribed into the corresponding expression in the q-deformed algebra. Thus the spectrum and eigenstates are immediately available from the representation theory of the q-deformed case, and moreover a one parameter fit to data is possible by adjusting the value of q.

Finally, a more indirect approach to quantum dynamical symmetries is available through the study of the real forms of the q-deformed algebras and the related question of embeddings of quantum algebras. This is obviously of paramount importance if the q-deformed algebras are to be considered as generalized spacetime symmetries; however it is also relevant to the study of dynamical (spectrum-generating) symmetries which are (for ordinary Lie algebras [15]) in many problems identical to familiar spacetime symmetries, with the emergence of non-compact real forms directly related to the infinite-dimensional nature of the spectrum of states to be encompassed by their unitary representations.

The thesis of the present paper is that it is most reasonable to look for q-deformations of the *radial* problem, while leaving the *angular* properties unchanged. Thus the procedure alluded to above, of the *ad hoc* replacement of a dynamical symmetry algebra by its q-deformation [14, 12, 10] may not be justified if in the application envisaged, there is still full rotational symmetry. In the case of the simple harmonic oscillator, for example, a Hamiltonian which admits an $U_q(\mathrm{su}(n))$ degeneracy symmetry [10] (for n degrees of freedom) is not appropriate if there is no suitable undeformed rotational subalgebra.

In section 2 below, we follow Bracken and Leemon [16] in analysing the algebraic structure of the D-dimensional isotropic simple harmonic oscillator. Scalar ladder operators for the radial quantum number can be defined having the character of modified bosons [17]. Together with suitably normalized angular momentum shift operators they realize the dynamical algebra $\mathrm{so}(2,1)\oplus\mathrm{so}(D,2)$. Consistently with the above discussion, it is suggested that a q-deformation allowing at least the dynamical symmetry $U_q(\mathrm{so}(2,1))\oplus\mathrm{so}(D)$, is appropriate for the *radial* q-deformation.

In section 3 this construction is carried out in detail for the two-dimensional case, following the algebraic analysis of [18]. A doubled space of commuting bosons is introduced for the radial sector, one set is q-deformed [6, 7, 9] the other is not, such that for suitable operators on the diagonal (equal occupation number) subspace, paying attention to rotational parity, the dynamical algebra $U_q(\mathrm{so}(2,1))\oplus\mathrm{so}(2)$ is realized.

Concluding remarks are given in section 4 together with comments concerning the possible q-differential realization of various alternative q-deformed Hamiltonians in the two-dimensional case.

2. The isotropic D-dimensional simple harmonic oscillator

In reference [16], the radial problem of the three dimensional simple harmonic oscillator was treated in a purely algebraic fashion for the first time. In this section, we shall briefly describe the way in which it is done, keeping in mind our objective of finding an appropriate deformation of part of the symmetry algebra of the problem. It was also noted in [18] that the construction can easily be extended to the case of $D\geqslant 3$ dimensions and it is this case which is sketched here.

In the coordinate representation of the D-dimensional simple harmonic oscillator, one considers a Hamiltonian

$$H = \hbar\omega\,(N + (D/2)) \tag{2.1}$$

where

$$N = N_1 + N_2 + \cdots + N_D \qquad N_i = a_i^\dagger a_i \quad i = 1, 2, \ldots D \tag{2.2}$$

$$a_j = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x_j + \mathrm{i}p_j\right) \tag{2.3}$$

$$a_j^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}\left(m\omega x_j - \mathrm{i}p_j\right) \tag{2.4}$$

$$\left[a_i, a_j^\dagger\right] = \delta_{i,j}. \tag{2.5}$$

One usually considers a basis of eigenstates of the number operators N_i (and hence the Hamiltonian H) of the form

$$|n_1, n_2, \ldots, n_D\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots n_D!}}(a_1^\dagger)^{n_1}(a_2^\dagger)^{n_2}\cdots(a_D^\dagger)^{n_D}|0\rangle \tag{2.6}$$

The bosonic operators $a_i^\dagger$ (respectively a_i) raise (lower) the eigenvalue of N_i. From the above remarks, these states are energy eigenstates with energy $\hbar\omega(n_1 + n_2 + \cdots + n_D + (D/2))$. This energy eigenvalue is, of course, very degenerate.

In fact, the Hermitian generators

$$(a_j a_k + a_j^\dagger a_k^\dagger) \qquad \mathrm{i}(a_j a_k - a_j^\dagger a_k^\dagger) \qquad (a_j^\dagger a_k + a_j a_k^\dagger) \tag{2.7}$$

together with the angular momentum generators (of the Lie algebra so(D))

$$L_{jk} = \frac{1}{\hbar}(x_j p_k - x_k p_j) = \mathrm{i}(a_j a_k^\dagger - a_k a_j^\dagger) \tag{2.8}$$

form a basis for the Lie algebra sp($2D, \mathbb{R}$). The Fock space decomposes into the direct sum of two irreducible (infinite-dimensional) representations of this algebra, one spanned by the eigenstates of the number operator N with even N-eigenvalue, the other spanned by those states with odd N-eigenvalue.

As an alternative, one can decompose the the total number operator N with respect to operators associated with the total angular momentum and the radial quantum number. What is more important from our point of view is that the appropriate spectrum generating algebra in this picture is not sp($2D, \mathbb{R}$) as in the case of the coordinate representation, but in fact so$(2,1) \oplus$ so$(D, 2)$. The quadratic Casimir of so(D) is well known to be $\frac{1}{2}L_{jk}L_{jk}$ and has an eigenvalue $l(l + D - 2)$ on some irreducible, highest weight representation, where l is a non-negative integer. One then defines the operator $L + (\frac{1}{2}D - 1)$ as the positive, scalar, Hermitian square root of the operator $\frac{1}{2}L_{jk}L_{jk} + (\frac{1}{2}D - 1)^2$. From this it follows that L has eigenvalues $l \geqslant 0$. Along with this we define an operator $K = \frac{1}{2}(N - L)$ (i.e. $N = 2K + L$) with eigenvalue $k \geqslant 0$, which turns out to be the radial quantum number.

Moreover, one can find (see the appendix) operators ν,$\nu^\dagger$ and λ, $\lambda^\dagger$ such that

$$K = \nu^\dagger \nu \qquad L = \lambda^\dagger \cdot \lambda \tag{2.9}$$

and $\nu^\dagger$ (respectively $\lambda^\dagger$) is a raising operator for K (L). That is we have the commutation relations

$$\left[L,\lambda_i^\dagger\right]=\lambda_i^\dagger \qquad [L,\nu]=0 \tag{2.10}$$

$$\left[K,\nu^\dagger\right]=\nu^\dagger \qquad [K,\lambda_i]=0 \tag{2.11}$$

as well as their conjugate relations. So instead of diagonalising the number operators N_i, one can diagonalize the operators K and L in the Fock space consisting of monomials in the creation operators $\nu^\dagger,\lambda_i^\dagger$ acting upon the vacuum state.

The generators of the dynamical symmetry algebra for the problem thus decomposed into angular and radial parts are

$$A_+=\tfrac{1}{2}(\nu^\dagger)^2 \qquad A_-=\tfrac{1}{2}(\nu)^2 \tag{2.12}$$

$$A_0=\tfrac{1}{2}(\nu^\dagger\nu+\tfrac{1}{2}) \tag{2.13}$$

which satisfy the $\mathrm{so}(2,1)\simeq\mathrm{sp}(2,\mathbb{R})\simeq\mathrm{sl}(2,\mathbb{R})$ relations

$$[A_0,A_\pm]=\pm A_\pm \qquad [A_+,A_-]=-2A_0 \tag{2.14}$$

while the angular momentum generators (of the Lie algebra $\mathrm{so}(D)$) are given in terms of λ as

$$L_{jk}\equiv \mathrm{i}(a_j a_k^\dagger-a_k a_j^\dagger)=-\mathrm{i}(\lambda_j^\dagger\lambda_k-\lambda_k^\dagger\lambda_j) \qquad 1\leqslant j,k\leqslant D. \tag{2.15}$$

Adjoining the further $2D+1$ generators $\lambda,\lambda^\dagger$ (suitably normalized) and $L+\frac{1}{2}(D-2)$, (see the appendix), the entire set of generators provide the spectrum generating algebra $\mathrm{so}(2,1)\oplus\mathrm{so}(D,2)$, in comparison with the coordinate representation, whose fundamental operators $a_i^\dagger, a_i$ furnished us with a realization of the Lie algebra $\mathrm{sp}(2D,\mathbb{R})$.

3. The 2D simple harmonic oscillator

As was noted in [19, 18] the situation when $D=2$ is somewhat different from the more general case for $D>2$. In particular there is a slight difficulty in correctly defining the operator L. The solution to this problem was given in [18]. If one defines the operator $M=L_{12}$ then one can consistently define the operator L as

$$L=|\,M\,| \tag{3.1}$$

and this provides L with the property that its eigenvalues are non-negative. Also the structure of the operators ν and λ become much simpler and these operators can, in fact, be expressed quite easily in terms of a pair of *angular* bosonic operators ρ and σ defined in terms of pair of common garden variety bosons via

$$\rho=\frac{1}{\sqrt{2}}(a_1-\mathrm{i}a_2) \tag{3.2}$$

$$\sigma=\frac{1}{\sqrt{2}}(a_1+\mathrm{i}a_2). \tag{3.3}$$

These angular bosons satisfy the relations

$$\left[\rho,\rho^{\dagger}\right]=1 \qquad \left[\sigma,\sigma^{\dagger}\right]=1 \tag{3.4}$$

$$[\rho,\sigma]=0 \qquad \left[\rho,\sigma^{\dagger}\right]=0. \tag{3.5}$$

Following [18] we shall now construct suitable operators ν and λ from which we will be able to construct a Fock space of states which will furnish a representation of the algebra $U_q(\mathrm{so}(2,1))\oplus\mathrm{so}(2)$. To do this we need two commuting sets of these *angular* bosons: one a normal pair, the other a q-deformed pair. Denote the undeformed bosons by ρ_1,σ_1 and the deformed bosons by ρ_2,σ_2. Let their corresponding number operators be $N^{\rho}_{\alpha},N^{\sigma}_{\alpha}$ $\alpha=1,2$. These operators satisfy the commutation relations

$$\left[\rho_1,\rho_1^{\dagger}\right]=1=\left[\sigma_1,\sigma_1^{\dagger}\right] \tag{3.6}$$

$$\rho_2\rho_2^{\dagger}-q\rho_2^{\dagger}\rho_2=q^{-N_2^{\rho}} \tag{3.7}$$

$$\sigma_2\sigma_2^{\dagger}-q\sigma_2^{\dagger}\sigma_2=q^{-N_2^{\sigma}} \tag{3.8}$$

$$[\rho_\alpha,\sigma_\alpha]=0=\left[\rho_\alpha,\sigma_\alpha^{\dagger}\right] \tag{3.9}$$

along with

$$\left[N^{\rho}_{\alpha},\rho_\alpha^{\dagger}\right]=\rho_\alpha^{\dagger}\;\left[N^{\sigma}_{\alpha},\rho_\alpha^{\dagger}\right]=0 \tag{3.10}$$

$$\left[N^{\sigma}_{\alpha},\sigma_\alpha^{\dagger}\right]=\sigma_\alpha^{\dagger}\;\left[N^{\rho}_{\alpha},\sigma_\alpha^{\dagger}\right]=0 \tag{3.11}$$

and relations conjugate to these. From these operators we can define a total number operator N_α, total angular momentum operator $L_\alpha=|\,M_\alpha\,|$ and hence the radial operator K_α by

$$N_\alpha=N^{\rho}_{\alpha}+N^{\sigma}_{\alpha} \qquad M_\alpha=N^{\rho}_{\alpha}-N^{\sigma}_{\alpha} \qquad K_\alpha=\tfrac{1}{2}(N_\alpha-L_\alpha). \tag{3.12}$$

Let $\mathcal{H}_\alpha$ be the Hilbert–Fock space of states in the monomials $\rho_\alpha,\sigma_\alpha$. A basis for the Fock spaces $\mathcal{H}_\alpha$ is provided by the vectors

$$|r,s\rangle_\alpha=K^{\alpha}_{rs}(\rho^{\dagger})^{r}(\sigma^{\dagger})^{s}|0\rangle \qquad r,s\geqslant 0 \tag{3.13}$$

where the normalization constants K^{α}_{rs} are given by

$$K^{1}_{rs}=(r!s!)^{-1/2} \tag{3.14}$$

$$K^{2}_{rs}=([r]_q![s]_q!)^{-1/2} \tag{3.15}$$

where

$$[x]_q=\frac{q^{x}-q^{-x}}{q-q^{-1}} \qquad [x]_q!=[x]_q[x-1]_q\cdots[2]_q[1]_q. \tag{3.16}$$

Let us now consider the space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. This space is spanned by the states

$$|r_1, s_1\rangle_1 \otimes |r_2, s_2\rangle_2 \qquad r_\alpha, s_\alpha \geqslant 0. \tag{3.17}$$

In particular we will be restricting our attention to the subspace $\mathcal{C}$ spanned by the vectors

$$|r, s\rangle = |r, s\rangle_1 \otimes |r, s\rangle_2. \tag{3.18}$$

This space splits up into the direct sum of subspaces

$$\mathcal{C} = \mathcal{C}_- \oplus \mathcal{C}_0 \oplus \mathcal{C}_+. \tag{3.19}$$

The space $\mathcal{C}_\pm$ is spanned by the states

$$|r, s\rangle \qquad \pm(r - s) > 0 \tag{3.20}$$

while $\mathcal{C}_0$ is spanned by the states

$$|r, r\rangle \qquad r \geqslant 0. \tag{3.21}$$

The operator M has positive (respectively zero, negative) eigenvalues on the space $\mathcal{C}_+$ (respectively $\mathcal{C}_0, \mathcal{C}_-$). Denote by P_0 (respectively $P_\pm$) the projection operator onto the space $\mathcal{C}_0$ (respectively $\mathcal{C}_\pm$).

Define operators on $\mathcal{C}$ as follows

$$N = \tfrac{1}{2}(N_1 \otimes 1 + 1 \otimes N_2) \tag{3.22}$$

$$K = \tfrac{1}{2}(K_1 \otimes 1 + 1 \otimes K_2) \tag{3.23}$$

$$M = \tfrac{1}{2}(M_1 \otimes 1 + 1 \otimes M_2) \tag{3.24}$$

$$L = \tfrac{1}{2}(L_1 \otimes 1 + 1 \otimes L_2). \tag{3.25}$$

Also define

$$\begin{aligned}
\nu &= (K_1+1)^{-1/2}(K_1+L_1+1)^{-1/2}\rho_1\sigma_1 \otimes [K_2+L_2+1]_q^{-1/2}\rho_2\sigma_2 \\
\nu^\dagger &= \rho_1^\dagger\sigma_1^\dagger(K_1+1)^{-1/2}(K_1+L_1+1)^{-1/2} \otimes \rho_2^\dagger\sigma_2^\dagger[K_2+L_2+1]_q^{-1/2} \\
\lambda &= \left\{\left(\frac{L_1+1}{K_1+L_1+1}\right)^{1/2} \otimes \frac{1}{[K_2+L_2+1]_q^{-1/2}}\right\}\{(\rho_1\otimes\rho_2)P_+ + (\sigma_1\otimes\sigma_2)P_-\} \\
\lambda^\dagger &= \left\{P_+(\rho_1^\dagger\otimes\rho_2^\dagger) + P_-(\sigma_1^\dagger\otimes\sigma_2^\dagger)\right\}\left\{\left(\frac{L_1+1}{K_1+L_1+1}\right)^{1/2} \otimes \frac{1}{[K_2+L_2+1]_q^{-1/2}}\right\}.
\end{aligned} \tag{3.26}$$

With calculations very similar to those performed in [18] one can verify that on the space $\mathcal{C}_+$, the matrix elements of the above operators are given by

$$\begin{aligned}
&N|r,s\rangle = (r+s)|r,s\rangle \quad K|r,s\rangle = s|r,s\rangle \quad L|r,s\rangle = (r-s)|r,s\rangle = M|r,s\rangle \\
&\nu|r,s\rangle = [s]_q^{1/2}|r-1,s-1\rangle \qquad \nu^\dagger|r,s\rangle = [s+1]_q^{1/2}|r+1,s+1\rangle \\
&\lambda|r,s\rangle = (r-s)^{1/2}|r-1,s\rangle \qquad \lambda^\dagger|r,s\rangle = (r-s+1)^{1/2}|r+1,s\rangle
\end{aligned} \tag{3.27}$$

Similarly, on the space $\mathcal{C}_-$ we have

$$N|r,s\rangle = (r+s)|r,s\rangle \quad K|r,s\rangle = r|r,s\rangle \quad L|r,s\rangle = (s-r)|r,s\rangle = -M|r,s\rangle$$

$$\nu|r,s\rangle = [r]_q^{1/2}|r-1,s-1\rangle \qquad \nu^\dagger|r,s\rangle = [r+1]_q^{1/2}|r+1,s+1\rangle \tag{3.28}$$

$$\lambda|r,s\rangle = (s-r)^{1/2}|r,s-1\rangle \qquad \lambda^\dagger|r,s\rangle = (s-r+1)^{1/2}|r,s+1\rangle$$

and on $\mathcal{C}_0$ we obtain

$$N|r,r\rangle = 2r|r,r\rangle \quad K|r,r\rangle = r|r,r\rangle \quad L|r,r\rangle = 0 = M|r,r\rangle$$

$$\nu|r,r\rangle = [r]_q^{1/2}|r-1,r-1\rangle \qquad \nu^\dagger|r,r\rangle = [r+1]_q^{1/2}|r+1,r+1\rangle \tag{3.29}$$

$$\lambda|r,r\rangle = 0 \qquad \lambda^\dagger|r,r\rangle = |r,r+1\rangle + |r+1,r\rangle.$$

Using the above actions of the operators we can confirm that, upon the space $\mathcal{C}$ the operators K, L, ν, λ satisfy

$$\left[K,\nu^\dagger\right] = \nu^\dagger \qquad \left[L,\lambda^\dagger\right] = \lambda^\dagger \tag{3.30}$$

$$[K,\nu] = -\nu \qquad [L,\lambda] = -\lambda \tag{3.31}$$

$$[K,\lambda] = 0 = \left[K,\lambda^\dagger\right] \tag{3.32}$$

$$[L,\nu] = 0 = \left[L,\nu^\dagger\right] \tag{3.33}$$

$$[\nu,\lambda] = 0 = \left[\nu,\lambda^\dagger\right] \tag{3.34}$$

$$\left[\nu^\dagger,\lambda\right] = 0 = \left[\nu^\dagger,\lambda^\dagger\right] \tag{3.35}$$

along with

$$\nu^\dagger\nu = [K]_q \qquad \nu\nu^\dagger = [K+1]_q \tag{3.36}$$

$$\lambda^\dagger\lambda = L+Q \qquad \lambda\lambda^\dagger = L+1+P_0 \tag{3.37}$$

where Q is an operator whose only non-zero action is

$$Q|r,r+1\rangle = |r+1,r\rangle \tag{3.38}$$

$$Q|r+1,r\rangle = |r,r+1\rangle. \tag{3.39}$$

So now we come to the question of what is the spectrum generating algebra for this system. Let us define operators $A_\pm, A_0$ acting on $\mathcal{C}$ by

$$A_+ = \frac{1}{\sqrt{q}(q+q^{-1})}(\nu^\dagger)^2 \tag{3.40}$$

$$A_- = \frac{1}{\sqrt{q}(q+q^{-1})}(\nu)^2 \tag{3.41}$$

$$A_0 = \tfrac{1}{2}K \tag{3.42}$$

Then it may be checked that these generators realize the algebra $U_{q^2}(\mathrm{so}(2,1))$ defined by

$$[A_0, A_\pm] = \pm A_\pm \tag{3.43}$$

$$[A_+, A_-] = -[2A_0]_{q^2}. \tag{3.44}$$

Thus the set of operators $\{A_\pm, A_0, L\}$ give rise to the spectrum generating algebra $U_{q^2}(\mathrm{so}(2,1)) \oplus \mathrm{so}(2)$.

4. Conclusion

We have discussed the question of q-deformations of the simple harmonic oscillator in quantum mechanics with emphasis on the physical requirement of an *undeformed* angular sector. An algebraic analysis of the radial problem following [16] in the $D = 3$ and [18] in the $D = 2$ case has been discussed with a view to q-deformation.

The analysis carried through for the $D = 2$ case involved the introduction of a doubled Fock space of oscillators, both deformed and undeformed [7] whose diagonal subspace provided the required representation of q-deformed laddering operators for the radial quantum number, and which commuted with the angular operators. The dynamical symmetry algebra was found to be $\mathrm{so}_q(2,1) \oplus \mathrm{so}(2)$. Although, *a posteriori*, one could start off defining the action of the operators λ, ν, K, L etc through equations (3.27)–(3.29), the introduction of the tensor product of Fock spaces provides us with a realization of the dynamical symmetry algebra, in terms of the fundamental bosonic and q-bosonic operators, along with the Hilbert space of states of the dynamical system.

In the general case we would expect the appropriate dynamical algebra to be $\mathrm{so}_q(2,1) \oplus \mathrm{so}(D,2)$ rather than a q-deformed version of the usual spectrum generating algebra $\mathrm{sp}(2D,\mathbb{R})$ arising from the coordinate representation in Cartesian coordinates. From the point of view of automorphisms of the q-deformed algebra and real forms, with undeformed rotational subalgebra [20], this suggests the need for a study of structures beyond the usual Chevalley or Cartan–Weyl basis.

In order to relate the algebraic discussion to a differential realization in the coordinate representation, one approach might be to employ the q-analogues of the eigenfunctions of the radial problem. Although this provides a readily applicable theory of q-orthogonal functions, the formulation of a Schrödinger equation [21] in the coordinate representation leading to the q-Laguerre functions as radial eigenfunctions founders on the technicalities of the behaviour of q-differentials in the separation of variables to angular and radial coordinates.

A more direct transcription would be to work from standard q-differential realizations of $\mathrm{so}_q(2,1)$ for the radial sector, together with standard spherical harmonics, in order to construct states as wavefunctions.

As to the form of the Hamiltonian operator, again the purely algebraic treatment does not give much insight. A natural choice for the radial sector, which matches the above comments concerning the nature of the wavefunctions, would for example be the q-deformed conformal supersymmetric quantum mechanics [22], which has been shown to admit $\mathrm{so}_{q^2}(2,1)$ as a dynamical symmetry. There the radial quantum number K is only formally defined on energy eigenstates, but has a natural differential realization in terms of the anticommutator of the factorization operators

$$A^\dagger = (1/\sqrt{2})(P_r - \mathrm{i}W(r))T_q \qquad A = q^{-D}T_{q^{-1}}(1/\sqrt{2})(P_r + \mathrm{i}W(r))$$

where $P_r = -\mathrm{i}(\mathrm{d}/\mathrm{d}r + (D-1/2r))$, and $T_q = q^{r\mathrm{d}/\mathrm{d}r}$; W(r) is a particular shape-invariant potential. Then formally

$$H_r = 2(q^{2K} - 1)/((q^2 - 1))$$

and the operators $\nu = qAq^{-K/2}$, $\nu^\dagger = q^{(1-K/2)}A^\dagger$, generate $\mathrm{so}_{q^2}(2,1)$ in the usual way. With this system we would have for the total Hamiltonian

$$H = \hbar\omega(H_r + L + (D/2))$$

which has the correct limit $2K + L + D/2$ as $q \to 1$.

The use of quantum enveloping algebras constitutes a marked liberalization of notions of symmetry in physical problems. The examination of q-deformations of dynamical symmetries of soluble problems is one aspect of this programme. Properly formulated q-analogues are likely to exhibit significant algebraic and geometrical features.

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Appendix. Derivation of the Lie algebra so$(D, 2)$

Here we sketch some details of the algebraic construction of the dynamical algebra $\mathrm{so}(2,1) \oplus \mathrm{so}(D,2)$ concentrating on the definition and closure relations of operators $\nu,\nu^\dagger$ and $\lambda,\lambda^\dagger$ considered in the text.

Firstly from (2.5) and the definition (2.8) follows

$$a_i L_{jk} + a_k L_{ij} + a_j L_{ki} = 0 \tag{A.1}$$

from which, using $\frac{1}{2} L_{kl} L_{kl} = L(L + D - 2)$,

$$a_i L_{ij} L_{jk} + a_i (D-2) L_{ik} + a_k L(L + D - 2) = 0 \tag{A.2}$$

or

$$a_i (L_{ij} - iL\delta_{ij})(L_{jk} + i(L + D - 2)\delta_{jk}) = 0 \tag{A.3}$$

where the commutation relations

$$[L_{ij}, L_{kl}] = -\mathrm{i}\left(\delta_{jk} L_{il} - \delta_{ik} L_{jl} - \delta_{jl} L_{ik} + \delta_{il} L_{jk}\right) \tag{A.4}$$

have been used. From (A.3) we may infer that a_i may be resolved into vector operators

$$\begin{aligned} a_j &= \left(a_i \left[(L + D - 2)\,\delta_{ij} - iL_{ij}\right] + a_i \left[L\delta_{ij} + iL_{ij}\right]\right)(2L + D - 2)^{-1} \\ &\equiv a_j^{(+)} + a_j^{(-)} \end{aligned} \tag{A.5}$$

and similarly for

$$a_j^\dagger = (a_j^\dagger)^{(+)} + (a_j^\dagger)^{(-)} \tag{A.6}$$

which shift L by ± 1.

Since from (A.5)

$$a_i^{(-)}(2L+D-2) = -a_i^\dagger(a\cdot a) + a_i(L+N+D-2)$$
$$(a_i^\dagger)^{(+)}(2L+D-2) = -a_i(a^\dagger\cdot a^\dagger) + a_i^\dagger(L+N+D-2)$$

we have $(a_i^{(\pm)})^\dagger = (a_i^\dagger)^{(\mp)}$ and so one can focus on one set, say $a_i^{(-)}$ and $(a_i^{(-)})^\dagger$, as defining fundamental shift operators λ_i and $\lambda_i^\dagger$ after suitable normalization, while the scalar combinations $a^\dagger\cdot a^\dagger$, $a\cdot a$ shift $K=\frac{1}{2}(N-L)$ by ± 1 and become the radial operators $\nu^\dagger$ and ν after normalization.

In considering the dynamical spectrum-generating algebra generated by $\nu^\dagger$, ν and $\lambda_i^\dagger$, λ_i it is important to verify the closure under commutation. For example

$$\begin{aligned}
a_i^{(-)}&(2L+D-2)a_j^{(-)}(2L+D-2)\\
&= \left(a_i(L+N+D-2) - a_i^\dagger(a\cdot a)\right)a_j^{(-)}(2L+D-2)\\
&= -a_i^\dagger(a\cdot a)a_j^{(-)}(2L+D-2) + a_i a_j^{(-)}(2L+D-2)(L+N+D-4)\\
&= -a_i^\dagger(a\cdot a)\left[-a_j^\dagger(a\cdot a) + a_j(L+N+D-2)\right]\\
&\quad + a_i(a\cdot a)\left[-a_j^\dagger(a\cdot a) + a_j(L+N+D-2)\right](L+N+D-4)\\
&= a_i^\dagger a_j^\dagger(a\cdot a)^2 - (a_i^\dagger a_j + a_j^\dagger a_i)(a\cdot a)(L+N+D-4)\\
&\quad + a_i a_j(L+N+D-2)(L+N+D-4)
\end{aligned}$$

is manifestly symmetric, so that

$$[\lambda_i, \lambda_j] = 0 \tag{A.7}$$

and similarly

$$\left[\lambda_i^\dagger, \lambda_j^\dagger\right] = 0. \tag{A.8}$$

On the other hand, the combination $\lambda_i\lambda_j^\dagger$ commutes with L, and in fact the commutator (with λ_i,$\lambda_j^\dagger$ suitably normalized) closes on L_{ij} and $(L+\frac{1}{2}(D-2))\delta_{ij}$. Thus, as in the case [16] for $D=3$, the $2D+1$ generators λ_i,$\lambda_i^\dagger$ and $L+\frac{1}{2}(D-2)$ can be appended to L_{ij} to form the Lie algebra of so$(D,2)$.

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